An extension of a theorem of Kesten to topological Markov chains

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Abstract. The main results of this note extend a theorem of Kesten for symmetric random walks on discrete groups to group extensions of topological Markov chains. In contrast to the result in probability theory, there is a notable asymmetry in the assumptions on the base. That is, it turns out that, under very mild assumptions on the continuity and symmetry of the associated potential, amenability of the group implies that the Gurevič-pressures of the extension and the base coincide whereas the converse holds true if the potential is Hölder continuous and the topological Markov chain has big images and preimages. Finally, an application to periodic hyperbolic manifolds is given.

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1 Introduction and statement of main results

The motivation for the analysis of the change of pressure under group extensions stems from the attempt to relate two classical results from probability theory and geometry on the amenability of discrete groups. The probabilistic result was obtained by Kesten in [11] and characterises amenability in terms of the spectral radius of the Markov operator associated to a symmetric random walk, that is a group G is amenable if and only if the spectral radius of the operator acting on $\ell^2(G)$ is equal to 1. The following counterpart in geometry was discovered by Brooks ([3]) using a completely different method. Assume that G is a Kleinian group acting on hyperbolic space $\mathbb{H}^{n+1}$ with exponent of convergence $\delta(G)$ bigger than $n/2$ and that $N \lhd G$ is a normal subgroup. Then the bottoms of the spectra of the Laplacians on $\mathbb{H}/G$ and $\mathbb{H}/N$ are equal if and only if G/N is amenable. Or

equivalently, using the characterisation of the bottom of the spectrum in terms of the exponents of convergence, G/N is amenable if and only if $\delta(G) = \delta(N)$. More recently, these results were partially improved. Roblin ([15]) used conformal densities to prove that amenability implies $\delta(G) = \delta(N)$ if G is of divergence type and Sharp obtained in [19] the same statement for convex-cocompact Schottky groups using Grigorchuk's results on the co-growth of shortest representations (see [7]) applied to the Cayley graph of G.

In here, we consider group extensions of a topological Markov chain for a given potential function. That is, for a topological Markov chain (Σ_A, θ), a potential $\varphi : \Sigma_A \to (0,\infty)$ and a map $\psi : \Sigma_A \to H$ from Σ_A to a discrete group H, the group extension of $(\Sigma_A, \theta, \varphi)$ by ψ is defined by

$$T : \Sigma_A \times H \to \Sigma_A \times H, (x,g) \mapsto (\theta(x), g\psi(x))$$

and the lifted potential by $\hat{\varphi} : \Sigma_A \times H \to \mathbb{R}, (x,g) \mapsto \varphi(x)$, where it is throughout assumed that ψ is constant on the states $\mathcal{W}^1$ of Σ_A. This then gives rise to a natural notion of symmetry through the existence of an involution $\kappa : \mathcal{W}^1 \to \mathcal{W}^1$ such that $\psi([\kappa w]) = \psi([w])^{-1}$ for all states $w \in \mathcal{W}^1$, where $[w]$ refers to the cylinder associated to w. This involution then extends to finite words, leading to the notion of a weakly symmetric potential by requiring that there exists a sequence (D_n) with $\lim_n D_n^{1/n} = 1$ such that

$$\sup_{x \in [w], y \in [\kappa w]} \frac{\prod_{j=0}^{n-1} (\varphi \circ \theta^j(x))}{\prod_{j=0}^{n-1} (\varphi \circ \theta^j(x))} \leq D_n,$$

for all $w \in \mathcal{W}^n$ with $\mathcal{W}^n$ referring to the words of length n (see Section 3 for details). Note that this general framework establishes a connection between random walks on groups and the geodesic flow on the unit tangent bundle of $\mathbb{H}/N$ for a certain class of Kleinian groups G, since random walks can be recovered by assuming that Σ_A is a symmetric full shift equipped with a locally constant, symmetric potential whereas the relation to the geodesic flow is obtained through a group extension of the coding map associated with G as considered, e.g., in [1] or [12].

The main results in here extend Kesten's result for random walks to group extensions by replacing statements on the spectral radius by statements on the Gurevič pressure P_G. Furthermore, they reveal a certain asymmetry with respect to the method of proof and the requirements on the mixing properties of the base transformation θ. The first result, Theorem 4.1, essentially states that, if the potential is weakly symmetric and the group H is amenable, then $P_G(T) = P_G(\theta)$. This result is an immediate corollary of Kesten's result, since the assumptions give rise to a construction of self-adjoint operators P_n on $\ell^2(H)$, for each $n \in \mathbb{N}$, whose spectral radii have to be equal to $\exp(nP_G(\theta))$ as a consequence of Kesten's theorem and the amenability of H. The arguments for obtaining the converse statement in Theorem 5.4 are more intricate and require that the potential is Hölder continuous and summable and that θ has the big images and preimages property. In this situation, we then have that $P_G(T) = P_G(\theta)$ implies that H is amenable. The proof is inspired by an argument of Day in [4] and relies on a careful analysis of the action of the Ruelle operator on the embedding of $\ell^2(H)$ into a certain subspace of $C(\Sigma_A \times H)$. As an application to hyperbolic geometry we then obtain that the theorem of Brooks extends

to the class of essentially free Kleinian groups which, in particular, may contain parabolic elements of arbitrary rank.

While writing this article, related results were independently obtained by Jaerisch ([8]) using a version of Kesten's result for graphs in [14]. In there, the assumptions in Theorems 4.1 and 5.4 were derived assuming that φ is locally constant, weakly symmetric and Σ_A is a subshift of finite type. The crucial difference to the methods in here relies on the assumption on the potential, since it appears to be impossible to adapt the method in there to general Hölder potentials.

2 Topological Markov chains

For a countable alphabet I and a matrix $A = (a_{ij} : i,j \in I)$ with $a_{ij} \in \{0,1\}$ for all $i,j \in I$ and $\sum_j a_{ij} > 0$ for all $i \in I$, let the pair (Σ_A, θ) denote the associated one-sided topological Markov chain. That is,

$$\Sigma_A := \left\{(w_k : k = 0,1,\ldots) : w_k \in I, a_{w_k w_{k+1}} = 1 \ \forall i = 0,1,\ldots\right\},$$
$$\theta : \Sigma_A \to \Sigma_A, \theta : (w_k : k = 1,2,\ldots) \mapsto (w_k : k = 2,3,\ldots).$$

A finite sequence $w = (w_1 \ldots w_n)$ with $n \in \mathbb{N}$, $w_k \in I$ for $k = 1,2,\ldots,n$ and $a_{w_k w_{k+1}} = 1$ for $k = 1,2,\ldots,n-1$ is referred to as a *word of length* n, and the set

$$[w] := \{(v_k) \in \Sigma_A : w_k = v_k \ \forall k = 1,2,\ldots,n\}$$

as a *cylinder of length* n. The *set of admissible words of length* n will be denoted by $\mathcal{W}^n$, the length of $w \in \mathcal{W}^n$ by $|w|$ and the set of all admissible words by $\mathcal{W}^\infty = \bigcup_n \mathcal{W}^n$. Furthermore, since $\theta^n : [w] \to \theta^n([w])$ is a homeomorphism, the inverse exists and will be denoted by $\tau_w : \theta^n([w]) \to [w]$. For $a,b \in \mathcal{W}^\infty$ and $n \in \mathbb{N}$ with $n \geq \max\{|a|,|b|\}$, set

$$\mathcal{W}^n_{a,b} = \{(w_1 \cdots w_n) \in \mathcal{W}^n : (w_1 \ldots w_{|a|}) = a, \ w_n b \text{ admissible}\}.$$

As it is well known, Σ_A is a Polish space with respect to the topology generated by cylinders and Σ_A is compact and locally compact with respect to this topology if and only if I is a finite set. Furthermore, recall that Σ_A is called *topologically transitive* if for all $a,b \in I$, there exists $n_{a,b} \in \mathbb{N}$ such that $\mathcal{W}^{n_{a,b}}_{a,b} \neq \emptyset$ and that Σ_A is called *topologically mixing* if for all $a,b \in I$, there exists $N_{a,b} \in \mathbb{N}$ such that $\mathcal{W}^n_{a,b} \neq \emptyset$ for all $n \geq N_{a,b}$. Moreover, a topological Markov chain is said to have *big images* or *big preimages* if there exists a finite set $\mathcal{I}_{\text{bip}} \subset \mathcal{W}$ such that for all $v \in \mathcal{W}$, there exists $\beta \in \mathcal{I}_{\text{bip}}$ such that $(v\beta) \in \mathcal{W}^2$ or $(\beta v) \in \mathcal{W}^2$, respectively. Finally, a topological Markov chain is said to have the *big images and preimages (b.i.p.) property* if the chain is topologically mixing and has big images and preimages (see [17]). Note that the b.i.p. property coincides with the notion of finite irreducibility for topological mixing topological Markov chains as introduced by Mauldin and Urbanski ([13]).

We now consider a pair (Σ_A, φ) where $\varphi : \Sigma_A \to \mathbb{R}$ is a strictly positive function which we refer to as a *potential*. For $n \in \mathbb{N}$ and $w \in \mathcal{W}^n$, set

$$\Phi_n := \prod_{k=0}^{n-1} \varphi \circ \theta^k \text{ and } C_w := \sup_{x,y \in [w]} \Phi_n(x)/\Phi_n(y). \tag{1}$$

The potential φ is said to have *(locally) bounded variation* if φ is continuous and there exists $C > 0$ such that $C_w \le C$ for all $n \in \mathbb{N}$ and $w \in \mathcal{W}^n$, and is called potential of *medium variation* if φ is continuous and, for all $n \in \mathbb{N}$, there exists $C_n > 0$ with $C_w \le C_n$ for all $w \in \mathcal{W}^n$ and $\lim_{n\to\infty} \sqrt[n]{C_n} = 1$. For positive sequences $(a_n),(b_n)$ we frequently will write $a_n \ll b_n$ if there exists $C > 0$ with $a_n \le C b_n$ for all $n \in \mathbb{N}$, and $a_n \asymp b_n$ if $a_n \ll b_n \ll a_n$. A further, stronger assumption on the variation is related to local Hölder continuity. Therefore, recall that the n-th variation of a function $f : \Sigma_A \to \mathbb{R}$ is defined by

$$V_n(f) = \sup\{|f(x) - f(y)| : x_i = y_i,\ i = 1,2,\dots,n\}.$$

The function f is referred to as a *locally Hölder continuous function* if there exists $0 < r < 1$ and $C \ge 1$ such that $V_n(f) \ll r^n$ for all $n \ge 1$. Moreover, we refer to a locally Hölder continuous function with $\|f\|_\infty < \infty$ as a Hölder continuous function. We now recall the following well-known estimate. For $n \le m$, $x, y \in [w]$ for some $w \in \mathcal{W}^m$, and a locally Hölder continous function f,

$$\Big|\sum_{k=0}^{n-1} f \circ \theta^k(x) - f \circ \theta^k(y)\Big| \ll \frac{1}{1-r} r^{m-n}. \tag{2}$$

In particular, the function $\exp f$ is a potential of bounded variation. For a given potential φ, the basic objects of thermodynamic formalism are partition functions. Since the state space might be countable, we consider partition functions Z_a^n for a fixed $a \in I$ which are defined by

$$Z_a^n := \sum_{\theta^n(x)=x,\, x\in[a]} \Phi_n(x).$$

Furthermore, we refer to the exponential growth rate of Z_a^n, that is to

$$P_G(\theta,\varphi) := \limsup_{n\to\infty} \log \sqrt[n]{Z_a^n} = \limsup_{n\to\infty} \frac{1}{n} \log Z_a^n,$$

as the *Gurevič pressure* of $(\Sigma_A, \theta, \varphi)$. This notion was introduced in [16] for topologically mixing systems where $\log\varphi$ is locally Hölder continuous. If $(\Sigma_A, \theta, \varphi)$ is transitive and φ is of medium variation, arguments in there combined with the decomposition of θ^p into mixing components, where p stands for the period of (Σ_A, θ), show that $P_G(\theta, \varphi)$ is independent of the choice of a and that

$$P_G(\theta,\varphi) = \lim_{n\to\infty,\, \mathcal{W}^n_{a,a}\neq\emptyset} \frac{1}{n} \log Z_a^n.$$

Furthermore, it is easy to see that $P_G(\theta,\varphi)$ remains unchanged by replacing $a \in \mathcal{W}^1$ with some $a \in \mathcal{W}^n$. Also recall that, if $\log\varphi$ is Hölder continuous and the system is topologically mixing, a variational principle holds (see [16]).

We now recall the definitions of conformal and Gibbs measures related to a given potential φ. A Borel probability measure μ is called *φ-conformal* if

$$\mu(\theta(A)) = \int_A \frac{1}{\varphi} d\mu$$

for all Borel sets A such that $\theta|_A$ is injective. For $(w_1 \dots w_{n+1}) \in \mathscr{W}^{n+1}$ and a potential of medium variation, it then immediately follows that

$$C_n^{-1}\mu(\theta([w_{n+1}])) \le \frac{\mu([w_1 \dots w_{n+1}])}{\Phi_n(x)} \le C_n \mu(\theta([w_{n+1}])) \tag{3}$$

for all $x \in [w_1 \dots w_{n+1}]$. Note that this estimate implies that $P_G(\theta, \varphi) = 0$ is a necessary condition for the existence of a conformal measure with respect to a potential of medium variation. Moreover, the above estimate motivates the following definitions. Assume that there exists a sequence $(B_n : n \in \mathbb{N})$ with $B_n \ge 1$ such that

$$B_n^{-1} \le \frac{\mu([w])}{\Phi_n(x)} < B_n \tag{4}$$

for all $n \in \mathbb{N}$, $w \in \mathscr{W}^n$ and $x \in [w]$. If $\sup_n B_n < \infty$, then μ is called *φ-Gibbs measure*, and if $\lim_{n\to\infty} B_n^{1/n} = 1$, then μ is called *weak φ-Gibbs measure*. A further fundamental object for our analysis is the Ruelle operator which is defined by, for $f : \Sigma_A \to \mathbb{R}$,

$$L_\varphi(f) = \sum_{v \in \mathscr{W}} \varphi \circ \tau_v \cdot f \circ \tau_v.$$

3 Group extensions of topological Markov chains

To introduce the basic object of our analysis, fix a discrete group G and a map $\psi : \Sigma_A \to G$ such that ψ is constant on $[w]$ for all $w \in \mathscr{W}^1$. Then, with $X := \Sigma_A \times G$ equipped with the product topology, the *group extension or G-extension* (X, T) of (Σ_A, θ) is defined by

$$T : X \to X, (x, g) \mapsto (\theta x, g\psi(x)).$$

Note that (X, T) is a topological Markov chain with cylinder sets $[w, g] := [w] \times \{g\}$, for $w \in \mathscr{W}^\infty$ and $g \in G$. Furthermore, set $X_g := \Sigma_A \times \{g\}$ and

$$\psi_n(x) := \psi(x)\psi(\theta x) \cdots \psi(\theta^{n-1} x)$$

for $n \in \mathbb{N}$ and $x \in \Sigma_A$. Observe that $\psi_n : \Sigma_A \to G$ is constant on cylinders of length n and, in particular, that $\psi(w) := \psi_n(x)$, for some $x \in [w]$ and $w \in \mathscr{W}^n$, is well defined. Moreover, for $a, b \in \mathscr{W}^1$ and $n \in \mathbb{N}$, let

$$G_n(a, b) := \{\psi(w) : n \in \mathbb{N}, w \in \mathscr{W}^n, [a] \supset [w], \theta^n([w]) \supset [b]\}.$$

Note that (X, T) is topologically transitive if and only if, for $a, b \in \mathscr{W}^1$, $g \in G$, there exists $n \in \mathbb{N}$ with $g \in G_n(a, b)$ and that (T, X) is *topologically mixing* if and only if, for $a, b \in \mathscr{W}^1$ and $g \in G$, there exists $N \in \mathbb{N}$ (depending on a, b, g) such that $g \in G_n(a, b)$ for all $n > N$. Note that the base transformation (θ, Σ_A) of a topologically transitive group extension has to be topologically transitive, and topologically mixing if the extension is topologically mixing, respectively. Furthermore, if (T, X) is topologically transitive, then $\{\psi(a) : a \in \mathscr{W}^1\}$ is a generating set for G.

Throughout, we now fix a topological mixing topological Markov chain (Σ_A, θ), and a topological transitive G-extension (T, X). Furthermore, we fix a (positive) potential $\varphi : \Sigma_A \to \mathbb{R}$ with $P_G(\theta, \varphi) = 0$. Note that φ lifts to a potential φ^* on X by setting $\varphi^*(x, g) := \varphi(x)$. For ease of notation, we will not distinguish between φ^* and φ. Moreover, for $v \in \mathscr{W}^\infty$, the inverse branch given by $[v, \cdot]$ will be as well denoted by τ_v, that is $\tau_v(x, g) := (\tau_v(x), g\psi(v)^{-1})$. In order to distinguish between the Ruelle operator of θ and T, these objects for the group extension will be written in calligraphic letters, that is, for $a \in \mathscr{W}$, $\xi \in [a] \times \{\mathrm{id}\}$, $(\eta, g) \in X$, and $n \in \mathbb{N}$,

$$\mathscr{L}(f)(\xi, g) := \sum_{v \in \mathscr{W}} \varphi(\tau_v(\xi)) f \circ \tau_v(\xi, g).$$

4 Extensions by amenable groups

In this section, we show that the Gurevič pressure remains unchanged under extension by an amenable group. In particular, it will turn out that this statement is true under very mild conditions. Also note that a similar result was proven in [19] for extensions of subshifts of finite type with respect to a hyperbolic potential. We first recall the definition of amenability using Følner's condition (see [5]). That is, G is referred to as an *amenable group* if and only if there exists a sequence (K_n) of finite subsets of G with $K_n \nearrow G$ such that

$$\lim_{n \to \infty} |gK_n \triangle K_n| / |K_n| = 0 \quad \forall g \in G.$$

In here, $\triangle$ refers to the symmetric difference, and $|\cdot|$ to the cardinality of a set. We are now interested in the characterisation of amenability in terms of the Gurevič pressure of a group extension. As it will turn out, this is an extension of the following result of Kesten (see [11]). Let m be a probability measure on G with $m(g^{-1}) = m(g)$ and assume that the support $\mathrm{supp}(m)$ of m is a generating set for G. Then the spectral radius of the operator P on the complex space $\ell^2(G)$ given by $Pf(\gamma) := \sum_{g \in G} f(\gamma g^{-1}) m(g)$ is equal to one if and only if G is amenable.

In order to obtain an extension of this result to shift spaces, one has to consider group extensions with symmetry. Namely, we say that (Σ_A, θ, ψ) is *symmetric* if there exists an involution $\kappa : \mathscr{W}^1 \to \mathscr{W}^1$, that is $\kappa \circ \kappa = \mathrm{id}$, with the following properties.

(i) For $v, w \in \mathscr{W}^1$, the word (vw) is admissible if and only if $(\kappa w \kappa v)$ is admissible.

(ii) $\psi(x) = \psi(y)^{-1}$ for all $v \in \mathscr{W}^1$, $x \in [v]$, $y \in [\kappa v]$.

Moreover, we refer to φ as a *weakly symmetric potential* if φ is continuous and there exists a sequence (D_n) with $\lim_{n \to \infty} \sqrt[n]{D_n} = 1$ such that, for all $n \in \mathbb{N}$ and $w \in \mathscr{W}^n$,

$$\sup_{x \in [w], y \in [\kappa w]} \Phi_n(x) / \Phi_n(y) \leq D_n.$$

Furthermore, if $\sup D_n < \infty$, then φ is referred to as a *symmetric potential*. Note that a weakly symmetric or symmetric potential is necessarily of medium variation with respect to $C_n := D_n^2$ or of bounded variation, respectively.

We now prove that amenability of G implies that $P_G(T) = P_G(\theta)$. Therefore, we construct a family of self-adjoint operators on $\ell^2(G)$ for symmetric group extensions as follows. For $(v_1 v_2 \dots v_n) \in \mathcal{W}^n$, set $\kappa(v_1 v_2 \dots v_n) = (\kappa v_n \dots \kappa v_2 \kappa v_1)$. In particular, this extends κ to an involution of $\mathcal{W}^\infty$. Now, by transitivity, there exists $a \in \mathcal{W}^\infty$ with $\psi(a) = \mathrm{id}$. This then gives rise to a further involution ι on $\mathcal{W}^n_{a,\kappa a}$ with the following symmetries. Since, for $n \geq |a|$, $(a v_1 \cdots v_{n-|a|}) \in \mathcal{W}^n_{a,\kappa a}$ if and only if $(a \kappa v_{n-|a|} \cdots \kappa v_1) \in \mathcal{W}^n_{a,\kappa a}$ we have that $\iota(a v_1 \cdots v_{n-|a|}) := (a \kappa v_{n-|a|} \cdots \kappa v_1)$ defines an involution on $\mathcal{W}^n_{a,\kappa a}$. Furthermore, for $v \in \mathcal{W}^n_{a,\kappa a}$, it follows that $\psi(v) = \psi(\iota v)^{-1}$.

Now assume that $P_G(\theta)$ is finite and fix $\xi \in [\kappa a]$. For $n \in \mathbb{N}$ and $v \in \mathcal{W}^n_{a,\kappa a}$, set $\pi_v := \frac{1}{2}(\Phi_n(\tau_v(\xi)) + \Phi_n(\tau_{\iota v}(\xi)))$. This then gives rise to an operator $P_n : \ell^2(G) \to \ell^2(G)$ on the complex Hilbert space $\ell^2(G)$ by

$$P_n(f)(\gamma) := \sum_{v \in \mathcal{W}^n_{a,\kappa a}} \pi_v f(\gamma \psi(v)^{-1}).$$

Note that $P_n(1) = L^n_\varphi(\mathbf{1}_{[a]})(\xi) < \infty$ and, with $\langle f, g \rangle = \sum \overline{f(\gamma)} g(\gamma)$ referring to the standard inner product, $\langle \mathbf{1}_\gamma, P_n(\mathbf{1}_{\gamma^*}) \rangle = \langle P_n(\mathbf{1}_\gamma), \mathbf{1}_{\gamma^*} \rangle$, for $\gamma, \gamma^* \in G$. In particular, this implies that P_n is self-adjoint.

Furthermore, combining the fact that $\langle \mathbf{1}_\gamma, P_n(\mathbf{1}_\gamma) \rangle = \langle \mathbf{1}_{\mathrm{id}}, P_n(\mathbf{1}_{\mathrm{id}}) \rangle$ for all $\gamma \in G$ with P_n being self-adjoint then gives that the spectral radius ρ_n of P_n satisfies (see, e.g., [9])

$$\rho_n = \limsup_{k \to \infty} \sqrt[k]{\langle \mathbf{1}_{\mathrm{id}}, P^k_n(\mathbf{1}_{\mathrm{id}}) \rangle},$$

and that, if P_n is a positive operator, then the lim sup above is a limit. As an immediate corollary to Kesten's theorem on the spectral radius of the Markov operator associated with a symmetric random walk (see [11] and [9, Theorem 5]), we obtain the following theorem for group extensions under very mild conditions.

Theorem 4.1. *Assume that T is a topologically transitive, symmetric group extension of the topologically mixing topological Markov chain (Σ_A, θ), φ is weakly symmetric and $P_G(\theta)$ is finite. Then $P_G(T) = P_G(\theta)$, if G is an amenable group.*

Proof. For $n \geq |a|$ and $g \in G$, set

$$m_n(g) := \frac{1}{P_n(1)} \sum_{v \in \mathcal{W}^n_{a,\iota a} : \psi(v) = g} \pi_v.$$

Note that $m_n(g)$ is well defined since $|P_G(\theta)| < \infty$ and that m_n is a symmetric probability measure on G, that is $m_n(g) = m_n(g^{-1})$. Moreover, note that the group generated by the support of m_n is a subgroup of G and hence is amenable. Since the Markov operator associated with the symmetric random walk given by m_n coincides with $P_n(\cdot)/P_n(1)$ we have by Kesten's theorem that $\rho_n = P_n(1)$. In order to prove the assertion it hence remains to show that $\lim_n \log(\rho_n)/n = P_G(\theta)$ and $\limsup_n \log(\rho_n)/n \leq P_G(T)$.

Step 1. Choose $k \geq |a|$ and $v \in \mathcal{W}^k_{a,\kappa a}$. Then, for $n \geq k$,

$$\rho_n = P_n(1) = \sum_{v \in \mathcal{W}^n_{a,\kappa a}} \Phi_n(\tau_v(\xi)) \geq C_k^{-1} C_n^{-1} \Phi_k(\tau_v(\xi)) Z_b^{n-k},$$

where $b = \kappa a_{|a|}$ with $a = (a_1 \cdots |a|)$. Hence, $\limsup_n (\log \rho_n)/n = P_G(\theta)$.

Step 2. For ease of notation, we will write $x = y^{\pm k} z$ for $y^{-k} z \le x \le y^k z$. Then, for $w \in \mathscr{W}^n_{a,\kappa a}$, weak symmetry implies

$$\Phi_n(\tau_w(\xi)) = \Phi_{n+|a|}(\tau_w(\xi))/\Phi_{|a|}(\xi) = (C_n D_{n+|a|})^{\pm 1} \Phi_n(\tau_{\iota w}(\xi)).$$

Hence, there exists $(\widehat{D}_n)$ with $\pi_w = \Phi_n(\tau_w(\xi)) \widehat{D}_n^{\pm 1}$ and $\lim_n \widehat{D}_n^{1/n} = 1$. By transitivity, there exist $l \in \mathbb{N}$, and $v \in \mathscr{W}^l_{\kappa a a}$ with $\psi(v) = \mathrm{id}$. Hence, for $w_1, w_2, \ldots w_k \in \mathscr{W}^n_{a\kappa a}$, we have $w^* := (w_1 v w_2 v \cdots w_k v) \in \mathscr{W}^{k(n+l)}_{aa}$ and, for $x \in [w^*]$,

$$\Phi_{k(n+l)}(x) \ge \left(\widehat{D}_n^{-1} \inf\{\Phi_l(y) : y \in [va]\}\right)^k \prod_{i=1}^k \pi_{w_i}.$$

This then gives rise to the estimate

$$\frac{1}{nk} \log \langle 1_{\mathrm{id}}, P_n^k(1_{\mathrm{id}})\rangle \le \frac{1}{n} \log\left(\widehat{D}_n^{-1} \inf\{\Phi_l(y) : y \in [va]\}\right) + \frac{1}{nk} \log \mathscr{Z}^{k(n+l)}_{a,\mathrm{id}}.$$

Since P_n^2 is a positive operator with $\|P_n^2\| = \rho_n^2$, we obtain, by taking the limit for $k \to \infty$, $k \in 2\mathbb{N}$ and then for $n \to \infty$ that $\lim_n \log(\rho_n)/n \le P_G(T)$. □

5 Kesten's theorem for group extensions

The essential ingredient of the proof of Theorem 4.1 is the fact that a symmetric probability measure defines a symmetric operator on $\ell^2(G)$. So, in order to prove the analogue of Kesten's result for group extensions of topological Markov, it remains to show that $P_G(\theta) = P_G(T)$ implies amenability, where one is tempted again to use the spectral radius formula applied to symmetric operators on $\ell^2(G)$ as, e.g., in [9, p. 478]. However, it will turn out that the key step in here is to carefully analise an embedding of $\ell^2(G)$ and use an argument based on uniform rotundity to show that in case of amenable groups, almost eigenfunctions are indicator functions.

The arguments of proof in here rely on stronger topological mixing properties in the base. That is, we will have to assume that the base has the b.i.p.-property. This property then gives rise to the existence of the following finite subset of $\mathscr{W}^\infty$.

Lemma 5.1. *Assume that (X,T) is a topologically transitive group extension of (Σ_A, θ) and that (Σ_A, θ) has the b.i.p.-property. Then there exists $n \in \mathbb{N}$ and a finite subset $\mathscr{J}$ of $\mathscr{W}^n$ such that for each pair (β, β') with $\beta, \beta' \in \mathscr{I}_{bip}$ there exists $w_{\beta,\beta'} \in \mathscr{J}$ such that $(w_{\beta,\beta'}) \in \mathscr{W}^n$ and $\psi_n(w_{\beta,\beta'}) = \mathrm{id}$.*

Proof. Let $p \in \mathbb{N}$ refer to the period p of the transitive topological Markov chain (X,T). Then, for each $a \in \mathscr{W} \times G$, there exists $N_a \in \mathbb{N}$ such that $T^{ps}([a,g]) \supset [a,g]$ for all $s \ge N_a$ and $g \in G$. Hence, for each pair $(\beta, \beta') \in \mathscr{I}^2_{\mathrm{bip}}$ with $(\beta'\beta)$ admissible, there is $N_{\beta,\beta'}$ such that for each $s \ge N_{\beta,\beta'}$ there exists $v_{\beta,\beta'} \in \mathscr{W}^{ps-2}$ such that $(\beta'\beta v_{\beta,\beta'}\beta')$ is admissible and $\psi_{ps}(\beta'\beta v_{\beta,\beta'}) = \psi_{ps}(\beta v_{\beta,\beta'}\beta') = \mathrm{id}$. Since $\mathscr{I}_{\mathrm{bip}}$ is finite it follows that there exists k (given by $\max\{pN_{\beta,\beta'} : (\beta,\beta') \in \mathscr{I}^2_{\mathrm{bip}}\}$) such that $v_{\beta,\beta'}$ can be always chosen to be an element of $\mathscr{W}^{k-2}$.

By possibly adding finitely many states we may assume without loss of generality that the subsystem of Σ_A with alphabet $\mathscr{I}_{\text{bip}}$ is topologically mixing. It then follows from this that there exists some $l \in \mathbb{N}$ such that each pair (β_0, β_l) in $\mathscr{I}_{\text{bip}}$ can be connected by an admissible word of the form

$$w_{\beta_0,\beta_l} := (\beta_0 v_{\beta_0,\beta_1}\beta_1\beta_2 v_{\beta_2,\beta_3}\beta_3\beta_4 \dots v_{\beta_{l-1},\beta_l}\beta_l).$$

The assertion then follows with $\mathscr{J} := \{w_{\beta,\beta'} : \beta, \beta' \in \mathscr{I}_{\text{bip}}\}$. □

A further important consequence of the b.i.p.-property is the existence of an invariant measure. That is, if (Σ_A, θ) and φ are given such that (Σ_A, θ) has the b.i.p.-property, $\log\varphi$ is Hölder continuous and $\|L_\varphi(1)\|_\infty < \infty$, then there exists a $\exp(-P_G(\theta,\varphi))\cdot\varphi$-Gibbs measure μ and a Hölder-continuous eigenfunction h such that $h d\mu$ is an invariant probability measure. It is also worth noting that (Σ_A, θ) has the Gibbs-Markov property with respect to μ (see [13, 17]) as implicitely introduced in [2]. Moreover, since the function $\log h$ is uniformly bounded from above and below, we assume from now, on without loss of generality, that $P_G(\theta,\varphi) = 0$ and $L_\varphi(1) = 1$.

The existence of an invariant probability measure μ then gives rise to a family of Markov operators $\{Q_n : n \in \mathbb{N}\}$ on $\ell^2(G)$ defined by

$$Q_n(f)(\gamma) := \sum_{v \in \mathscr{W}^n} \mu([v]) f(g\psi(v)^{-1}).$$

We will also make use of a refined analysis of the spectrum of the operator $\mathscr{L}_\varphi$ on a certain space $\mathscr{H}$. For $g \in G$ and $f : X \to \mathbb{R}$, set $\|f\|_\infty^g := \sup\{|f(x,g)| : x \in \Sigma_A\}$ and

$$[\![f]\!]^2 := \sum_{g \in G} (\|f\|_\infty^g)^2 \quad \text{and} \quad \mathscr{H} := \{f : X \to \mathbb{R} : [\![f]\!] < \infty\}.$$

Furthermore, set $\mathscr{H}_c := \{f \in \mathscr{H} : f|_{X_g} \text{ const.} \forall g \in G\}$ and $\rho := \exp(P_G(T,\varphi))$.

Proposition 5.2. *The operator $\mathscr{L}_\varphi$ acts on the Banach space $(\mathscr{H}, [\![\cdot]\!])$ as a bounded operator and there exists $C \geq 1$ with $\rho^k \leq [\![\mathscr{L}_\varphi^k]\!] \leq C$ for all $k \in \mathbb{N}$. Furthermore,*

$$[\![\mathscr{L}_\varphi^k]\!] = \sup\left(\left\{[\![\mathscr{L}_\varphi^k(f)]\!]/[\![f]\!] : f \geq 0, f \in \mathscr{H}_c\right\}\right). \tag{5}$$

Proof. The proof of the assertion that $(\mathscr{H}, \|\cdot\|_{\mathscr{H}})$ is a Banach space is standard and therefore omitted. In order to prove the remaining assertion, we derive an estimate for $\|\mathscr{L}_\varphi^k(f)\|_{\mathscr{H}}^2$ where $k \in \mathbb{N}$ is arbitrary. Using Jensen's inequality, we obtain

$$\begin{aligned}
[\![\mathscr{L}_\varphi^k(f)]\!]^2 &\leq \sum_{g \in G} \sup_{x \in \Sigma_A} \left(\sum_{v \in \mathscr{W}^k} \Phi_k \circ \tau_v(x) \|f\|_\infty^{g\psi_k(v)^{-1}}\right)^2 \\
&\leq \sum_{g \in G} \sup_{x \in \Sigma_A} \sum_{v \in \mathscr{W}^k} \Phi_k \circ \tau_v(x) \left(\|f\|_\infty^{g\psi_k(v)^{-1}}\right)^2 \\
&\leq C \sum_{v \in \mathscr{W}^k} \mu([v]) [\![f]\!]^2 = C[\![f]\!]^2,
\end{aligned}$$

where C is given by the Gibbs property of μ on Σ_A (see (4)). Hence, C is an upper bound for $[\![\mathscr{L}_\varphi^k]\!]$, independent from k. Moreover, the identity in (5) follows from the first inequality in the estimate above and $[\![f]\!] = [\![\|f\|_\infty^\bullet]\!]$. In order to obtain the lower bound by ρ^n, note that $\mathbf{1}_{X_{\mathrm{id}}} \in \mathscr{H}$ and hence, by Hölder continuity of φ, there exists $C' > 0$ with

$$\|\mathscr{L}^n(\mathbf{1}_{X_{\mathrm{id}}})\|_\infty^{\mathrm{id}} \geq C' \sum_{\substack{T^n(x,\mathrm{id})=(x,\mathrm{id}),\\ x\in[a]}} \Phi_n(x)$$

for all $a \in \mathscr{W}$ and $n \in \mathbb{N}$. The remaining assertion follows from $[\![\mathscr{L}_\varphi^{kl}]\!] \leq [\![\mathscr{L}_\varphi^k]\!]^l$. □

The proof of the following result is inspired by an argument of Day (see [4, Lemma 4]) which relies on the rotundity of $\ell^2(G)$. Recall that a Banach space $(B, \|\cdot\|)$ is uniformly rotund if for all $\delta > 0$ there exists $\epsilon > 0$ such that, for all f, g with $\|f - g\| \geq \delta$ and $\|f\| = \|g\| = 1$, it follows that $\|f + g\| \leq 2 - \epsilon$. Note that the space $\mathscr{H}$ does not has this property but the closed subspace $\mathscr{H}_c$ which is isomorphic to $\ell^2(G)$. Since $\ell^2(G)$ is uniformly rotund, it follows that $\mathscr{H}_c$ has this property as well.

Lemma 5.3. *Assume that $\rho = 1$. Then there exists $n \in \mathbb{N}$ such that, for given $\epsilon > 0$, there exists $f \in \mathscr{H}_c$ with $f \geq 0$ and $[\![\mathscr{L}_\varphi^n(f) - f]\!] \leq \epsilon[\![f]\!]$.*

Proof. Let k be given by $\mathscr{J} \subset \mathscr{W}^k$ where $\mathscr{J}$ is the finite set given by Lemma 5.1. The first step is to prove that, if (f_m) is a sequence of positive functions in $\mathscr{H}_c$ with $[\![f_m]\!] = 1$ and $\lim_m [\![\mathscr{L}_\varphi^k(f_m)]\!] = 1$, then $\liminf_m [\![\mathscr{L}_\varphi^k(f_m) - f_m]\!] = 0$. So assume that

$$\liminf [\![\mathscr{L}_\varphi^k(f_m) - f_m]\!] =: \delta > 0.$$

It is then possible to choose a finite set $\mathscr{W}^* \subset \mathscr{W}^k$ such that

$$\sum_{v\in\mathscr{W}^k\setminus\mathscr{W}^*} \Phi_k \circ \tau_v(x) \leq \delta/(4C|\mathscr{I}_{\mathrm{bip}}|),$$

where C is given by bounded distortion, that is $C = \sup\{C_w : w \in \mathscr{W}^\infty\}$ (see (1)). It then follows, for $f \in \mathscr{H}_c$, that

$$\begin{aligned}
[\![\sum_{v\in\mathscr{W}^k\setminus\mathscr{W}^*} \Phi_k \circ \tau_v f \circ \tau_v]\!] &= \| \sup_{x\in X.} \sum_{v\in\mathscr{W}^k\setminus\mathscr{W}^*} \Phi_k \circ \tau_v(x) f \circ \tau_v(x,\cdot)\|_2 \\
&\leq C \sum_{a\in\mathscr{I}_{\mathrm{bip}}} \| \sup_{x\in[a,\cdot]} \sum_{v\in\mathscr{W}^k\setminus\mathscr{W}^*} \Phi_k \circ \tau_v(x) f \circ \tau_v(x,\cdot)\|_2 \leq \delta[\![f]\!]/4.
\end{aligned}$$

Using $L_\varphi(1) = 1$ and the $\triangle$-inequality then gives, for each $m \in \mathbb{N}$,

$$\begin{aligned}
\delta &\leq [\![\mathscr{L}_\varphi^k(f_m) - f_m]\!] = [\![\sum_{v\in\mathscr{W}^k} \Phi_k \circ \tau_v (f_m \circ \tau_v - f_m)]\!] \\
&\leq \sum_{v\in\mathscr{W}^*} [\![\Phi_k \circ \tau_v (f_m \circ \tau_v - f_m)]\!] + [\![\sum_{v\notin\mathscr{W}^*} \Phi_k \circ \tau_v f_m \circ \tau_v]\!] + [\![\sum_{v\notin\mathscr{W}^*} \Phi_k \circ \tau_v f_m]\!] \\
&\leq \sum_{v\in\mathscr{W}^*} \sup_{x\in\Sigma_A} (\Phi_k \circ \tau_v(x)) [\![(f_m \circ \tau_v - f_m)\mathbf{1}_{\theta^k([v])\times G}]\!] + \delta/2.
\end{aligned}$$

Hence, with $D := \sum_v \|\Phi_k \circ \tau_v(x)\|_\infty < \infty$, it follows from a convex sum argument, that there exists $v_m \in \mathscr{W}^*$ with $[\![(f_m \circ \tau_{v_m} - f_m)\mathbf{1}_{\theta^k([v_m])\times G}]\!] \geq \delta/(2D)$. Moreover, for each $a \in \mathscr{W}$, note that there exists $v_a \in \mathscr{J}$ such that $(v_a a)$ is admissible, and that $\psi(v_a) = \mathrm{id}$ implies that $f(x) = f \circ \tau_{v_a}(x)$ for all $x \in [a]$ and $f \in \mathscr{H}_c$. Furthermore, since $\mathscr{J}$ and $\mathscr{W}^*$ are finite, $\Phi_k \circ \tau_w(x)$ is uniformly bounded away from 0 and 1 for all $w \in \mathscr{J} \cup \mathscr{W}^*$ and $x \in \theta^k([w])$. Hence, the uniform rotundity of $\mathscr{H}_c$ implies that there exists $\hat{\delta} > 0$ with

$$\left\| \frac{\Phi_k \circ \tau_{v_m}(x) f_m \circ \tau_{v_m}(x,\cdot) + \Phi_k \circ \tau_{v_a}(x) f_m \circ \tau_{v_a}(x,\cdot)}{\Phi_k \circ \tau_{v_m}(x) + \Phi_k \circ \tau_{v_a}(x)} \right\|_2 \leq 1 - \hat{\delta}$$

for all $m \in \mathbb{N}$ and $x \in \theta^k([v_m]) \cap [a]$. In particular,

$$\|\mathscr{L}^k(f_m)(x,\cdot)\|_2 \leq 1 - \hat{\delta}(\Phi_k \circ \tau_{v_m}(x) + \Phi_k \circ \tau_{u_x}(x)).$$

By substituting x with $\tau_w(y)$ for some arbitrary $y \in \Sigma_A$ and $w \in \mathscr{J}$, it follows from $\psi_k(w) = \mathrm{id}$ and $|\mathscr{J}| < \infty$ that there exists $D' > 0$ with $[\![\mathscr{L}_\varphi^{2k}(f_m)]\!] \leq 1 - D'\hat{\delta}$ for all $m \in \mathbb{N}$. This proves the statement above.

Finally, recall from Proposition 5.2 that there exists a sequence of functions $(f_m : m \in \mathbb{N})$ in $\mathscr{H}_c$ with $[\![f_m]\!] = 1$ and $[\![\mathscr{L}_\varphi^{2k}(f_m)]\!] \nearrow 1$. The assertion of the Lemma follows from this for $n = 2k$. □

This now enables us to obtain a converse to Theorem 4.1. Note that in here, it is not required that neither the group extension nor the potential are symmetric.

Theorem 5.4. *Assume that (Σ_A, θ) is a topological Markov chain with big images and preimages (b.i.p.), that (X,T) is a topologically transitive G-extension and that φ is a Hölder continuous potential with $\|L_\varphi(1)\|_\infty < \infty$. Then $P_G(T,\varphi) = P_G(\theta,\varphi)$ implies that the group G is amenable.*

Proof. We assume without loss of generality that $P_G(\theta,\varphi) = 0$. Now choose a finite subset K of G. It then follows by decomposition of T into mixing components that there exists $m \in \mathbb{N}$ such that m is a multiple of n in Lemma 5.3 and $K \subset \{\psi_m(v) : v \in \mathscr{W}^m\}$. In particular, there exists a finite subset $\mathscr{W}_K$ of $\mathscr{W}^m$ with

$$K = \{\psi_m(v) : v \in \mathscr{W}_K\}.$$

By applying Lemma 5.3, it then follows that there exists a sequence (f_k) in $\mathscr{H}_c$ such that $\lim_{k\to\infty} [\![\mathscr{L}^m(f_k) - f_k]\!] = 0$, $f_k \geq 0$ and $[\![f_k]\!] = 1$ for all $k \in \mathbb{N}$. Using the same argument as in the proof of the Lemma, it follows that

$$\lim_{k\to\infty} [\![(f_k \circ \tau_v - f_k) \cdot \mathbf{1}_{[\theta^m(v),\cdot]}]\!] = 0, \quad \forall v \in \mathscr{W}_K.$$

For $f \in \mathscr{H}_c$, let $\hat{f} \in \ell^2(G)$ be given by $\hat{f}(g) := f(x,g)$ for some $x \in \Sigma_A$. Then, using the Hölder inequality, we have, for $h \in G$,

$$\begin{aligned} \|\hat{f}_k^{\,2}(\cdot) - \hat{f}_k^{\,2}(\cdot h)\|_1 &= \sum_{g\in G} |\hat{f}_k^{\,2}(g) - \hat{f}_k^{\,2}(gh)| = \sum_{g\in G} |\hat{f}_k(g) - \hat{f}_k(gh)| \cdot |\hat{f}_k(g) + \hat{f}_k(gh)| \\ &\leq \|\hat{f}_k(\cdot) - \hat{f}_k(\cdot h)\|_2 \cdot \|\hat{f}_k(\cdot) + \hat{f}_k(\cdot h)\|_2 \leq 2\|\hat{f}_k(\cdot) - \hat{f}_k(\cdot h)\|_2 \end{aligned} \quad (6)$$

We now fix k to be specified later and use the following representation of $\hat{f_k}^2$. There exist $p \in \mathbb{N} \cup \{\infty\}$ and $\lambda_i > 0$ and $A_i \subset G$ with $A_i \subset A_{i+1}$, for $1 \le i < p$, such that $\hat{f_k}^2 = \sum_{i=1}^p \lambda_i \mathbf{1}_{A_i}$. In particular, observe that $\sum_{i=1}^p \lambda_i |A_i| = 1$ and that, by monotonicity of (A_i) we always have that $(A_i h \setminus A_i) \cap (A_j \setminus A_j h) = \emptyset$. Hence,

$$\begin{aligned} \|\hat{f_k}^2(\cdot) - \hat{f_k}^2(\cdot h)\|_1 &= \sum_{g \in G} \left| \sum_i \lambda_i (\mathbf{1}_{A_i h^{-1}}(g) - \mathbf{1}_{A_i}(g)) \right| \\ &= \sum_{g \in G} \sum_{i=1}^p \lambda_i \mathbf{1}_{A_i h^{-1} \triangle A_i}(g) = \sum_{i=1}^p \lambda_i |A_i h^{-1} \triangle A_i| \end{aligned} \tag{7}$$

We are now in position to prove amenability of G. For $\epsilon > 0$, choose k such that

$$[\![(f_k \circ \tau_v - f_k) \cdot \mathbf{1}_{[\theta^m(v),\cdot]}]\!] = \|\hat{f}_k(\cdot) - \hat{f}_k(\cdot \psi_m(v)^{-1})\|_2 \le \epsilon / |\mathscr{W}_K|$$

for all $v \in \mathscr{W}_K$. Combining estimate (6) and the identity (7) then implies that

$$\epsilon \ge \frac{1}{2} \sum_{i=1}^p \lambda_i \sum_{h \in K} |A_i h \triangle A_i|.$$

Hence, there exists $1 \le i < p$ with $\sum_{h \in K} |A_i h \triangle A_i| \ge 2\epsilon |A_i|$. □

It is worth noting that the proof is inspired by an argument which can be found in [6] where the identity (7) was used to derive a weak Følner condition. However, there is the following alternative chain of arguments for the proof. It follows from Lemma 5.3 that $\|Q_n\| = 1$. A theorem of Day then implies that G is amenable.

Finally, observe that there are the following immediate implications to Gibbs-Markov maps and the co-growth of groups. Assume that (Σ_A, θ, μ) is a topologically mixing Gibbs-Markov map with the b.i.p.-property and that (X, T) is a topologically transitive G-extension. It then follows from Theorem 5.4 that G is amenable if

$$\limsup_{n \to \infty} (\mu(\{x \in \Sigma_A : \psi_n(x) = \mathrm{id}\})^{1/n} = 1. \tag{8}$$

Furthermore, if the potential $d\mu/d\mu \circ \theta$ is weakly symmetric, then Theorem 4.1 implies that condition (8) is equivalent to amenability. Note that this might also be seen as a result in the context of random walks on groups with stationary increments.

As a further application we obtain a criteria for amenability in terms of the co-growth as introduced in [7]. For a set of generators $\mathscr{G} = \{\gamma_1, \gamma_1^{-1}, \ldots, \gamma_r^{-1}\}$ of G, let Σ_A be the subshift of finite type with $\mathscr{W} = \mathscr{G}$ and transition matrix (a_{gh}) given by $a_{gh} = 0$ if and only if $(g, h) = (\gamma_i, \gamma_1^{-1}), (\gamma_i^{-1}, \gamma_1)$ for some $i = 1, \ldots, r$. With respect to the potential 1 and the cocycle $\psi|_{[g]} = g$ we then obtain the Grigorchuk-Cohen result. That is, G is amenable if and only if

$$\limsup_{n \to \infty} |\{w \in \mathscr{W}^n : \psi_n(w) = \mathrm{id}\}|^{1/n} = |\mathscr{G}| - 1 = 2r - 1.$$

6 An application to hyperbolic geometry

In order to apply the above results to normal covers of hyperbolic manifolds, we recall the following definition from [20]. A Kleinian group is called essentially free if there exists a Poincaré fundamental polyhedron F with faces $f_1, f_2, \dots f_{2n}$ and associated generators $g_1, g_2 \dots g_{2n}$ of G with $g_i(f_i) = f_{i+n}$, $g_i^{-1}(f_{i+n}) = f_i$ and $g_i^{-1} = g_{i+n}$ for $i = 1, \dots n$, such that the following conditions are satisfied. In here, $\overline{(\cdot)}_{\overline{\mathbb{H}}}$ refers to the closure in $\overline{\mathbb{H}}$.

(i) If $\overline{(f_i)}_{\overline{\mathbb{H}}} \cap \overline{(\bigcup_{j \neq i} f_j)}_{\overline{\mathbb{H}}} \neq \emptyset$ for some $i = 1, 2, \dots n$, then g_i, g_{i+n} are hyberbolic transformations, and $\overline{(f_{i+n})}_{\overline{\mathbb{H}}} \cap \overline{(\bigcup_{j \neq i+n} f_j)}_{\overline{\mathbb{H}}} \neq \emptyset$,

(ii) if $\overline{(f_i)}_{\overline{\mathbb{H}}} \cap \overline{(f_j)}_{\overline{\mathbb{H}}}$ is a single point p for some $j = 1, 2, \dots 2n$, then p is a parabolic fixed point,

(iii) if $f_i \cap f_j \neq \emptyset$ for some $j = 1, 2, \dots 2n$, then $g_i g_j = g_j g_i$.

Observe that this class comprises all non-cocompact Fuchsian groups, the class of Schottky groups, and in general gives rise to geometrically finite hyperbolic manifolds which may have cusps of arbitrary rank.

We now proceed with the construction of the associated coding map. Fix a point $\mathbf{o}$ in the interior F, and denote by a_i the intersection of the shadow of f_i and the radial limit set $L_r(G)$ of G. Furthermore, denote by κ the inversion, acting on $\{f_i : i = 1, \dots 2n\}$, $\{g_i : i = 1, \dots 2n\}$ and $\{a_i : i = 1, \dots 2n\}$, defined by $\kappa f_i = f_{i+n}$, $\kappa g_i = g_{i+n}$ and $\kappa a_i = a_{i+n}$, respectively. Now let a be an atom of the partition α generated by $\{a_i : i = 1, \dots 2n\}$. Hence there exist $1 \leq k \leq 2n$ and $i_1, \dots i_k \in \{1, \dots 2n\}$ such that $a = \bigcap_{l=1}^{k} a_{i_l}$. Choose $g_a \in \{g_{i_l} : k = 1, \dots k\}$, and for $\kappa a := \bigcap_{l=1}^{k} \kappa a_{i_l}$, set $g_{\kappa a} = g_a^{-1}$. This then gives rise to the coding map Γ defined by

$$\Gamma : L_r(G) \to L_r(G), \theta(x) = g_a(x) \text{ for } x \in a, a \in \alpha.$$

For further details of this construction, we refer to [20], where it is shown, that θ is well defined, α is a Markov partition and the underlying subshift of finite type is topologically mixing. In particular, $L_r(G)$ can be identified with a shift space Σ_A and Γ with the one-sided shift map.

In order to specify a potential adapted to the geometry of $\mathbb{H}$, recall that the Poisson kernel $\mathcal{K}$ with respect to the ball model is given by, for $x \in \partial\mathbb{H}$ and $\mathbf{o} \in \mathbb{H}$,

$$\mathcal{K}(\mathbf{o}, x) := \frac{1 - |g(\mathbf{o})|^2}{|g(\mathbf{o}) - x|^2}.$$

It is well known that $\log \mathcal{K}(g(\mathbf{o}), x)$, for $g \in G$, is equal to the orientated hyperbolic distance between $\mathbf{o}$ and the horocycle through $g(\mathbf{o})$ and x. Note that this horocyclic distance sometimes also is referred to as the Busemann cocycle. Furthermore, we recall the following for the potential given by

$$\varphi(x) = (\mathcal{K}(g_a(\mathbf{o}), x))^{\delta}, \tag{9}$$

for $x \in a, a \in \alpha$ and $\delta > 0$ to be specified later. If G is convex cocompact, then $\log \varphi$ is Hölder continuous with respect to the shift metric induced by Σ_A. In case that G is essentially

free and contains parabolic elements, set

$$P := \overline{F} \cap \{p \in L(G) : p \text{ is a fixed point of a parabolic element in } G\}.$$

Then, if B is a subset of $L_r(G)$ which is bounded away from P and measurable with respect to some finite refinement by preimages of α, the potential associated with the first return map to B is Hölder continuous (see [10, Lemmata 3.3 & 3.4]). For ease of notation, let (Σ, θ) refer to the first return map to B if G contains parabolic elements and to (Σ_A, Γ) if G does not contain parabolic elements. Observe that in both cases the potential is Hölder continuous with respect to the shift metric, (Σ, θ) has the b.i.p.-property and can be identified with a maximal non-invertible factor of a Poincaré section of the geodesic flow on the unit tangent bundle $T^1(\mathbb{H}/G)$ of $\mathbb{H}/G$ (see [20]). Furthermore, since by construction B is bounded away from P, each return to the associated Poincaré section in $T^1(\mathbb{H}/G)$ corresponds to a return to a ball of bounded diameter with center $\mathbf{o}$ in $\mathbb{H}/G$.

Combining these observations, one then obtains the following relation between the finite words of Σ, the elements of G and the hyperbolic distance $d(\mathbf{o}, g(\mathbf{o}))$. Let $\mathscr{W}^n$ refer to the words of length n of Σ and $g_w \in G$ to the element in G defined by $\theta^n|_{[w]} = g_w$. Note that, by construction of Γ, G is isomorphic to the finite words with respect to the original partition α (see [20]). Hence, after a possible induction to B, the map $\mathscr{W}^\infty \to G$, $w \mapsto g_w$ is injective. Using the property of bounded returns, it follows that the map is almost onto in the sense that there exists a finite subset J of G such that

$$G = \bigcup_{h \in J, w \in \mathscr{W}^\infty} h g_w. \tag{10}$$

Then, for $w \in \mathscr{W}^\infty$, set $\Phi_w := \sup_{x \in [w]} \Phi_{|w|}(x)$. As a further consequence of bounded returns, it then follows that, for $s > 0$,

$$\Phi_w^s \asymp e^{-s\delta d(\mathbf{o}, g_w(\mathbf{o}))}. \tag{11}$$

Now assume that N is a normal subgroup of G and recall that in this situation, the manifold $\mathbb{H}/N$ is called periodic with period G/N. Since $\mathbb{H}/N$ is a cover of $\mathbb{H}/G$, it follows that $\mathbb{H}/N$ is geometrically finite if and only if G/N is finite. The properties (11) and (10) above now allow relating the exponents of convergence of N and G in terms of the amenability of G/N. Therefore, recall that the Dirichlet series

$$\mathscr{P}(H, s) := \sum_{g \in H} e^{-sd(\mathbf{o}, g(\mathbf{o}))}$$

is referred to as the Poincaré series of the Kleinian group H, and that its abzissa of convergence $\delta(H)$ is called the exponent of convergence of H. In order to quantify $\mathscr{P}(N, s)$, we will now employ our results on group extensions. Therefore, for $w \in \mathscr{W}$ and $x \in [w]$, set $\psi(x) := [g_w] \in G/N$ and

$$T : \Sigma \times G/N \to \Sigma \times G/N, \quad (x, [g]) \mapsto (\theta(x), [g]\psi(x)).$$

In here, we only will make use of the estimates on $\mathscr{L}^n(\mathbf{1}_{X_{\mathrm{id}}})$, but is worth noting that T is related to the geodesic flow on the periodic manifold. That is, T is a non-invertible factor of

the base transformation of a Ambrose-Kakutani representation of the flow on the periodic manifold (see also, e.g. [1, 18]), where the associated measure is the Liouville-Patterson measure of G.

As an application of Theorems 4.1 and 5.4 we now obtain the following partial refinement of a result of Brooks for convex-cocompact Kleinian groups in [3] to the class of essentially free Kleinian groups.

Theorem 6.1. *Let G be an essentially free Kleinian group and $N \lhd G$ a normal subgroup. Then $\delta(G) = \delta(N)$ if and only if G/N is amenable.*

Proof. Set $\delta = \delta(G)$ in the definition of φ and note that for this choice, the existence of a finite, invariant measure implies that $P_G(\theta, \varphi) = 0$ (see, e.g., [20]). Furthermore, as a consequence of (11) we have that φ is symmetric with respect to the involution generated by κ. Finally, it can easily be deduced from the connection between the group extension T and the geodesic flow on $T^1(\mathbb{H}/N)$ that T is topologically transitive. Hence, Theorems 4.1 and 5.4 are applicable and therefore, it remains to show that $P_G(T) < P_G(\theta)$ if and only if $\delta(N) < \delta = \delta(G)$.

So assume that $\delta(N) < \delta$. Hence, for $\epsilon > 0$ with $\delta(N) < (1-\epsilon)\delta$, $\mathscr{P}(N, (1-\epsilon)\delta) < \infty$. In particular, applying (10) and (11) gives that

$$\infty > \sum_{g \in N} e^{-(1-\epsilon)\delta(\mathbf{o}, g(\mathbf{o}))} \asymp \sum_{\substack{w \in \mathscr{W}^{\infty},\\ [g_w]=\mathrm{id}}} \Phi_w^{1-\epsilon} \geq \sum_{\substack{w \in \mathscr{W}^{\infty},\\ [g_w]=\mathrm{id}}} \Phi_w \|\varphi\|_{\infty}^{-\epsilon|w|}.$$

Since $\exp(-P_G(T))$ is the radius of convergence of $\sum_{[g_w]=\mathrm{id}} \Phi_w x^{|w|}$, it follows from $\|\varphi\|_\infty < 1$ that $P_G(T) \leq \epsilon \log \|\varphi\|_\infty < 0 = P_G(\theta)$.

Now assume that $P_G(T) < 0$ and G does not contain parabolic elements. Then $\mathscr{W}$ is finite and, in particular, $\|1/\varphi\|_\infty < \infty$. Therefore, since $P_G(T) < 0$, we have for $1 < x < \exp(-P_G(T))$ that

$$\begin{aligned} \infty > \sum_{\substack{w \in \mathscr{W}^{\infty},\\ [g_w]=\mathrm{id}}} \Phi_w x^{|w|} &\geq \sum_{\substack{w \in \mathscr{W}^{\infty},\\ [g_w]=\mathrm{id}}} \Phi_w^{1-\epsilon} x^{|w|} \|1/\varphi\|_\infty^{-\epsilon|w|} \\ &\asymp \sum_{\substack{w \in \mathscr{W}^{\infty},\\ [g_w]=\mathrm{id}}} e^{-(1-\epsilon)\delta(\mathbf{o}, g_w(\mathbf{o}))} x^{|w|} \|1/\varphi\|_\infty^{-\epsilon|w|}. \end{aligned} \tag{12}$$

Hence, if $\epsilon < -P_G(T)/\log(\|1/\varphi\|_\infty)$, then $\mathscr{P}(N, (1-\epsilon)\delta) < \infty$ and, in particular, $\delta(N) < \delta$.

It remains to consider the case where $P_G(T) < 0$ and G contains parabolic elements. For $p \in P$, let G_p denote the stabiliser of p in G. By well known arguments (see, e.g., Lemma 3.2 in [21]), we have, for $s > k_p/2$ and $\ell > 0$, with k_p referring to the Abelian rank of G_p, that

$$\sum_{\substack{g \in G_p,\\ d(\mathbf{o}, g(\mathbf{o})) \geq 2\ell}} e^{-sd(\mathbf{o}, g(\mathbf{o}))} \asymp \sum_{n \geq e^{\ell}} \frac{1}{n^{2s-k_p+1}} \asymp \frac{1}{2s-k_p} e^{\ell(k_p - 2s)}.$$

Since $\delta(G)$ is always bigger than $k_p/2$, we may choose $0 < \epsilon < 1 - k_p(2\delta)^{-1}$. For $s = (1-\epsilon)\delta$, we then have

$$\sum_{\substack{g \in G_p,\\ d(\mathbf{o}, g(\mathbf{o})) \geq 2\ell}} e^{-(1-\epsilon)\delta d(\mathbf{o}, g(\mathbf{o})} \asymp e^{2\ell\delta\epsilon} \sum_{\substack{g \in G_p,\\ d(\mathbf{o}, g(\mathbf{o})) \geq 2\ell}} e^{-\delta d(\mathbf{o}, g(\mathbf{o}))}.$$

For $\Lambda > 0$, set $\mathscr{W}_\Lambda := \{w \in \mathscr{W}^1 : \inf_{x\in[w]} \varphi(x) \geq \Lambda\}$. It follows from the inducing process of θ, that $g_w \in G_p$ for some $p \in P$ and each $w \in \mathscr{W}_\Lambda$, if $\Lambda > 0$ is sufficiently small. The above estimate then implies the following uniform Lipschitz continuity. There exists $C \geq 1$ such that for arbitrary families $\{x_w : x \in [w], w \in \mathscr{W}_\Lambda\}$,

$$\frac{\sum_{w\in\mathscr{W}_\Lambda} \varphi(x_w)^{1-\epsilon}}{\sum_{w\in\mathscr{W}_\Lambda} \varphi(x_w)} - 1 \leq C\epsilon.$$

Combining the estimate with the argument in (12) then gives that $\delta(N) < \delta(G)$. □

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